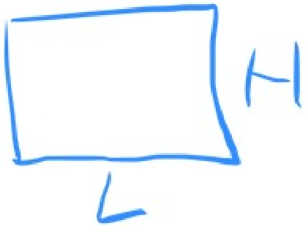
Last class

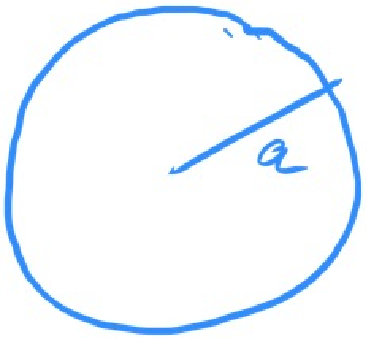
Laplace equation on a rectangle



$$u_{xx} + u_{yy} = 0$$

<u>today:</u>

Laplace equation on dish of radius a



<u>idea:</u>

use polar coordinates

Problem 1 Express Laplace operator

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

in polar coordinates!

$x = r\cos\theta$

$y = r\sin\theta$

Solution: in polar coordinates Laplace operator ∇^2 is given by

$$\nabla^2 u = \frac{1}{r}\,\partial/\partial r\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

this can be checked using chain rule as follows:

$f = f(x,y) = f(r\cos\theta, r\sin\theta)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\boxed{\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\boxed{\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta}$$

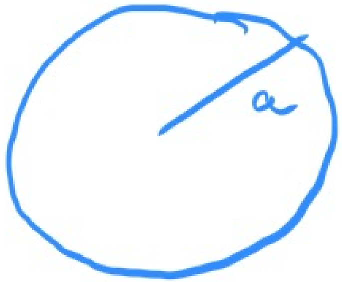
use this for $\frac{1}{r}\,\partial/\partial r\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$\Longrightarrow$ get back

$$\downarrow$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Statement of Problem:



solve the following problem:

$$\nabla^2 u = \frac{1}{r} \partial/\partial r \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

with boundary condition

$$u(a,\theta) = f(\theta)$$

given function

physical interpretation:

Find equilibrium temperature on a disk for prescribed temperatures $f(\theta)$ on boundary point $(a\cos\theta, a\sin\theta)$

Again try separation of variables:

$$u(r,\theta) = \phi(\theta)\, G(r)$$

plug into PDE:

$$\frac{1}{r}\,\partial/\partial r\Big(r\, \underbrace{\phi(\theta)G'(r)}_{\frac{\partial u}{\partial r}}\Big) + \frac{1}{r^2}\, \underbrace{\phi''(\theta)G(r)}_{\frac{\partial^2 u}{\partial \theta^2}} = 0$$

$$\frac{1}{r}\Big(\phi(\theta)G'(r) + r\,\phi(\theta)G''(r)\Big) + \quad '' \quad = 0$$

used product rule.

divide by $\phi(\theta)G(r)$

$$\frac{1}{r}\left(\frac{G'}{G} + r\,\frac{G''}{G}\right) + \frac{1}{r^2}\,\frac{\phi''}{\phi} = 0 \quad \Big|\, r^2$$

$$\underbrace{r\left(G'/G + r\,G''/G\right)}_{\text{only depends on } r} + \underbrace{\phi''/\phi}_{\text{only depends on } \phi} = 0$$

$$\Rightarrow \quad \frac{rG'}{G} + \frac{r^2G''}{G} = -\frac{\phi''}{\phi} = \lambda$$

get two ODE's

$$\boxed{\begin{aligned} \phi'' &= -\lambda \phi \\ r^2G'' + rG' &= \lambda G \end{aligned}}$$

additional conditions come from using polar coordinates!

θ and $\theta + 2\pi$ indicate same angle

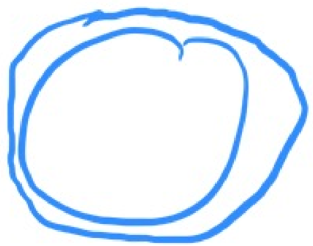
$$\Rightarrow \quad \boxed{\begin{aligned} \phi(-\pi) &= \phi(\pi) \\ \phi'(-\pi) &= \phi'(\pi) \end{aligned}} \quad \Leftarrow$$

problem solved last time for rectangle!

solution: $\lambda = n^2 \frac{\pi^2}{L^2} = n^2$ as $L = \pi$ here.

$$\boxed{\begin{aligned}\phi(\theta) &= c_1 \cos n\theta + c_2 \sin n\theta \\ &\text{for } \lambda = n^2,\ n > 0\end{aligned}}$$

why $L = \pi$? look back to problem of heat equation for ring



circumference $2L$

in this we had (almost) the same boundary cond. and ODE:

$$\left.\begin{aligned}\phi'' &= -\lambda \phi \\ \phi(-L) &= \phi(L) \\ \phi'(-L) &= \phi'(L)\end{aligned}\right\}$$

solution: $\lambda = \frac{n^2 \pi^2}{L^2}$

$$\phi(x) = c_1 \cos \frac{n\pi}{L} x + c_2 \sin \frac{n\pi}{L} x$$

$\Rightarrow$ our current problem for φ

= same as problem of :  of circumference 2π

$$\Rightarrow \cos\frac{n\pi}{L}x \longrightarrow \cos nx$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \rightarrow n^2$$

here we use variable θ

<u>new part</u>: have to solve

$$\boxed{r^2 G'' + rG' = n^2 G}$$

<u>$n>0$</u>: try $G(r) = r^\alpha$ for some exponent α.

$$G'(r) = \alpha r^{\alpha-1}$$

$$G''(r) = \alpha(\alpha-1) r^{\alpha-2}$$

$$\Rightarrow r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} = n^2 r^\alpha$$

$$\Rightarrow r^\alpha (\underbrace{\alpha(\alpha-1) + \alpha}_{=\alpha^2}) = n^2 r^\alpha$$

$$\Rightarrow r^{\alpha} \alpha^2 = n^2 r^{\alpha}$$

$$\Rightarrow \alpha^2 = n^2$$

$$\Rightarrow \alpha = \pm n$$

<u>Result:</u> The ODE $r^2 G'' + r G' = n^2 G$

has general solution

$$\boxed{G(r) = c_1 r^n + c_2 r^{-n}}$$

<u>If $n=0$:</u>

$$r^2 G'' + r G' = 0$$

$$\left.\begin{aligned} &\Rightarrow r G'' + G' = 0 \\ &\frac{d}{dr}(r G') = 0 \end{aligned}\right\} \Rightarrow r G' = c_1$$

$$\Rightarrow G' = \frac{c_1}{r}$$

$$\Rightarrow \boxed{G(r) = c_1 \log r + c_2}$$